11 Suppose U and W are both four-dimensional subspaces of $\mathbf{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

$$\dim(U) = \dim(W) = 4$$

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$$
$$\dim(U+W) = 8 - \dim(U \cap W) \quad — ①$$

Since, $(U+W)$ is subspace of $\mathbb{C}^6$. Then,
$$\dim(U+W) \leq \dim(\mathbb{C}^6) = 6 \quad — ②$$

By ① and ②, $2 \leq \dim(U \cap W)$.

Therefore, we can find two independent vectors in $U \cap V$ such that neither of these vectors is a scaler multiple of other.

12 Suppose that U and W are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $\mathbf{R}^8 = U \oplus W$.

We know that,

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

$$\dim(\mathbb{R}^8) = 8 = 5 + 3 - \dim(U \cap W)$$

$$0 = \dim(U \cap W)$$

Thus $(U \cap W) = \{0\}$. Therefore, $\mathbb{R}^8 = U \oplus W$

13 Suppose U and W are both five-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

$\dim(U) = \dim(W) = 5$.

Assume the contray $U \cap W = \{0\}$. Then $\dim(U \cap W) = 0$

Since U, W are subspaces of $\mathbb{R}^9$, then $U+W$ is a subspace of $\mathbb{R}^9$. Then

$$\dim(U+W) \leq \dim(\mathbb{R}^9) = 9 \text{ --- } (*)$$

$$9 \geq \dim(U+W) = \dim(U) + \dim(W) + \dim(U \cap W)$$
$$= 5 + 5 + 0$$

Thus, $9 \geq 10$.

This is a contradiction. Thus our hypothesis is wrong.

Therefore $U \cap W \neq \{0\}$

14 Suppose V is a ten-dimensional vector space and V_1, V_2, V_3 are subspaces of V with $\dim V_1 = \dim V_2 = \dim V_3 = 7$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

12
ප්‍රශ්නයේ අංකය මෙම තීරුවේ ලියන්න.
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Suppose that $\dim(V) = 10$.

Let V_1, V_2, V_3 are subspaces of V

with $\dim(V_1) = \dim(V_2) = \dim(V_3) = 7$

$\cancel{\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V)}$

By 2.43,

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$$

Then, $\dim(V_1 \cap V_2) = 7 + 7 - 10 = 4$ — (1)

Furthur, by 2.43,

$$\dim((V_1 \cap V_2) + V_3) = \dim(V_1 \cap V_2) + \dim(V_3) + \dim((V_1 \cap V_2) \cap V_3)$$

Furthur, by 2.43,

$$\dim((V_1 \cap V_2)+V_3) = \dim(V_1 \cap V_2) + \dim(V_3) + \dim((V_1 \cap V_2) \cap V_3)$$

$$\begin{aligned}\dim(V_1 \cap V_2 \cap V_3) &= \dim(V_1 \cap V_2) + \dim(V_3) - \dim((V_1 \cap V_2) + V_3) \\ &= 4 + 7 - \dim((V_1 \cap V_2) + V_3) \\ &= 13 - \dim((V_1 \cap V_2) + V_3) \\ &\geq 13 - \dim(V) = 13 - 10 = 3 \quad (*)\end{aligned}$$

~~For each $j = 1, 2, \ldots, m$~~

~~Choose a basis for V_j~~

~~Con~~ This is because, $(V_1 \cap V_2)$ and V_3 are subspaced of V and $(V_1 \cap V_2) + V_3$ is a subspaces of V. Then $\dim((V_1 \cap V_2) + V_3) \leq \dim(V) = 10$

By (*) $\dim(V_1 \cap V_2 \cap V_3) \geq 3$. Therefore $V_1 \cap V_2 \cap V_3 \neq \{0\}$

15 Suppose V is finite-dimensional and V_1, V_2, V_3 are subspaces of V with $\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.